SCHUR-WEYL DUALITY AND THE FREE LIE ALGEBRA

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Abstract. We prove an analogue of Schur-Weyl duality for the space of homogeneous Lie polynomials of degree r in n variables.

1. Introduction

Let k be a commutative ring and V a given k-module. Put $E = \operatorname{End}_k(V)$. The centralizer of a set $X \subseteq E$ of k-linear endomorphisms of V is the set

$$Z_E(X) = \{f \in E \mid fx = xf, \text{ for all } x \in X\}.$$

Suppose further that V has a given (A, B)-bimodule structure, where A and B are k-algebras. Let $\overline{A}$, $\overline{B} \subset \operatorname{End}_k(V)$ be the sets of k-linear endomorphisms of V induced by the actions of A and B, respectively. Since the actions of A and B commute, we have inclusions

$$\overline{A} \subseteq \operatorname{End}_B(V) \quad \text{and} \quad \overline{B} \subseteq \operatorname{End}_A(V), \tag{1}$$

where $\operatorname{End}_A(V) = Z_E(\overline{A})$ and $\operatorname{End}_B(V) = Z_E(\overline{B})$. When the inclusions in (1) are equalities then we say that the triple (A, V, B) satisfies Schur-Weyl duality. This implies that $\overline{A}$ and $\overline{B}$ both have the double centralizer property, that is,

$$Z_E(Z_E(\overline{A})) = \overline{A} \quad \text{and} \quad Z_E(Z_E(\overline{B})) = \overline{B}. \tag{2}$$

If $\overline{A}$ has the double centralizer property and $\overline{B} = Z_E(\overline{A}) = \operatorname{End}_A(V)$, then $\overline{A} = Z_E(\overline{B}) = \operatorname{End}_A(V)$ as well, and (A, V, B) satisfies Schur-Weyl duality. But the two equalities in (2) do not by themselves imply that (A, V, B) satisfies Schur-Weyl duality.

Assume henceforth that k is a field. An important example of the duality above is given by the r^{th} tensor power $V = T^r(k^n) = (k^n)^{\otimes r}$ of the space k^n of n-dimensional column vectors, regarded as an (A, B)-bimodule, where $A = k\operatorname{GL}_n(k)$ and $B = k\Sigma_r$ are respectively the group algebras of the general linear group $\operatorname{GL}_n(k)$ and symmetric group Σ_r, with $\operatorname{GL}_n(k)$ acting diagonally on the left and Σ_r acting on the right by place permutation. To be precise, the commuting actions are given by

$$g \cdot (v_1 \otimes \cdots \otimes v_r) = gv_1 \otimes \cdots \otimes gv_r \quad \text{and} \quad (v_1 \otimes \cdots \otimes v_r) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}$$

for all $v_1, \dots, v_r \in k^n$, $g \in \operatorname{GL}_n(k)$, and $\sigma \in \Sigma_r$. In this setting the assertion that the triple $(k\operatorname{GL}_n(k), T^r(k^n), k\Sigma_r)$ satisfies Schur-Weyl duality is the classical

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Schur-Weyl duality between representations of $\mathrm{GL}_n(k)$ and Σ_r, which is known to hold whenever $|k| > r$. (See [2] for a proof.)

In this note, we investigate the analogue of classical Schur-Weyl duality when tensor space $T^r(k^n)$ is replaced by its intersection $L^r(k^n)$ with the free Lie algebra $L(k^n)$ on k^n.

Recall that the free Lie algebra $L(k^n)$ is the Lie subalgebra of the tensor algebra $T(k^n)$ generated by k^n, where $T(k^n)$ is regarded as a Lie algebra via the Lie bracket $[a, b] = ab - ba$. Fixing a basis X of k^n, we may identify the tensor algebra $T(k^n)$ with the free associative algebra $k\langle X\rangle$ on X. In this point of view, elements of $k\langle X\rangle$ are regarded as noncommutative polynomials in "variables" X, and polynomials in the subspace $L(k^n)$ are known as Lie polynomials. The grading $T(k^n) = \bigoplus_{r\geq 0} T^r(k^n)$ induces a corresponding grading

$$L(k^n) = \textstyle\bigoplus_{r\geq 0} L^r(k^n), \quad \text{where} \quad L^r(k^n) = L(k^n) \cap T^r(k^n)$$

on $L(k^n)$. The rth graded component $L^r(k^n)$ in the above decomposition is the space of homogeneous Lie polynomials of degree r.

Since $g \cdot [a, b] = [g \cdot a, g \cdot b]$ for any $g \in \mathrm{GL}_n(k)$, $a, b \in k^n$, it is clear that $L(k^n)$ is invariant under the action of $\mathrm{GL}_n(k)$, hence is a left $k\,\mathrm{GL}_n(k)$-module. It follows that the same holds for $L^r(k^n)$. A natural problem is to describe the centralizer algebra $\mathrm{End}_{\mathrm{GL}_n(k)}(L^r(k^n))$ as a subquotient of $k\Sigma_r$; ideally one would like to identify a subalgebra B_r of $k\Sigma_r$ which maps onto this centralizer. Furthermore, having identified such a subalgebra B_r, it is natural to ask whether the triple $(k\,\mathrm{GL}_n(k), L^r(k^n), B_r)$ satisfies Schur-Weyl duality. When it does, we have an analogue of Schur-Weyl duality for the module $L^r(k^n)$ of homogeneous Lie polynomials of degree r.

Our main results explain how to identify the appropriate subalgebra B_r and establish that, indeed, $(k\,\mathrm{GL}_n(k), L^r(k^n), B_r)$ satisfies Schur-Weyl duality, provided only that the characteristic of k is strictly larger than r. (We agree that characteristic zero is infinite and hence always larger than any r.) Furthermore, under our assumption on the characteristic, it is well-known that an idempotent $e \in k\Sigma_r$ exists such that $L^r(k^n) = T^r(k^n)e$. Then B_r may be taken to be the subalgebra $ek\Sigma_r e$ of $k\Sigma_r$. The more general question, of whether Schur-Weyl duality holds whenever $|k| > r$ and the characteristic of k does not divide r, remains open.

The paper is organized as follows. In §2 we state the main theorem, Theorem 2.2, show that the triple $(k\,\mathrm{GL}_n(k), L^r(k^n), ek\Sigma_r e)$ satisfies Schur-Weyl duality when the characteristic of k is strictly larger than r, and draw some general conclusions.

The proof of Theorem 2.2 is given in §3 as an application of more general results. We consider a triple (A, V, B) that satisfies Schur-Weyl duality and an idempotent $e \in B$, and we ask when (A, Ve, eBe) satisfies Schur-Weyl duality. It turns out that $\overline{eBe}$ is always equal to $\mathrm{End}_A(Ve)$. The equality $\overline{A} = \mathrm{End}_{eBe}(Ve)$ seems to be a more delicate question. We show that if V is a completely reducible A-module whose irreducible constituents are absolutely irreducible, then in fact (A, Ve, eBe) satisfies Schur-Weyl duality.

In the case of classical Schur-Weyl duality, $L^r(k^n)$ is a tilting module and it is tempting to try to use the theory of tilting modules and a p-modular system to derive results in positive characteristic from known results in characteristic zero. This approach can be used to yield a uniform proof in all characteristics of some of

the known properties of the algebra $ek\Sigma_r e$, but we have been unable to show that $(k\,\mathrm{GL}_n(k), L^r(k^n), ek\Sigma_r e)$ satisfies Schur-Weyl duality in general.

Finally, in §4 we describe the commuting algebra $ek\Sigma_r e$ in the favorable case when the characteristic of k is strictly larger than r, and we show that in general, whether $(k\,\mathrm{GL}_n(k), L^r(k^n), ek\Sigma_r e)$ satisfies Schur-Weyl duality may be reduced to a statement about permutation representations of Σ_r arising from Young subgroups.

2. Notation and Main Results

In this section we establish our basic notation and formulate the main results.

Recall that k denotes a field. Throughout the paper, we set $T^{n,r} = T^r(k^n)$ and $L^{n,r} = L^r(k^n)$ for ease of notation, and we denote by Φ and Ψ the k-algebra homomorphisms

$$k\,\mathrm{GL}_n(k) \xrightarrow{\Phi} \mathrm{End}_k(T^{n,r}) \xleftarrow{\Psi} k\Sigma_r$$

induced by the commuting actions of $\mathrm{GL}_n(k)$ and Σ_r described in the introduction. Note that because Σ_r acts on the right, the homomorphism Ψ is given by $\Psi(\sigma)(x) = x \cdot \sigma^{-1}$ for $\sigma \in \Sigma_r$ and $x \in T^{n,r}$. Then classical Schur-Weyl duality is the pair of equalities

$$\Phi(k\,\mathrm{GL}_n(k)) = \mathrm{End}_{\Sigma_r}(T^{n,r}) \quad \text{and} \quad \Psi(k\Sigma_r) = \mathrm{End}_{\mathrm{GL}_n(k)}(T^{n,r}). \tag{3}$$

It will be convenient to define a Lie idempotent to be any idempotent e in $k\Sigma_r$ such that $L^{n,r} = T^{n,r} \cdot e$. This agrees with the usual definition (see e.g. [11, §8.4]) when $n \geq r$. Lie idempotents exist whenever the characteristic of k does not divide r. Proofs of the following well-known result may be found in [5, §2] and [11, §8.4].

Lemma 2.1. *Assume that the characteristic of k does not divide r. Then the Dynkin-Specht-Wever idempotent $e_r = \frac{1}{r}(1-\gamma_2)\cdots(1-\gamma_r)$, where for $2 \leq i \leq r$, γ_i is the descending i-cycle $(i\ \cdots\ 2\ 1)$ in Σ_r, is a Lie idempotent.*

Suppose that e is a Lie idempotent. Then $ek\Sigma_r e$ acts on $L^{n,r}$ on the right and $\mathrm{GL}_n(k)$ acts on $L^{n,r}$ on the left. Thus there are k-algebra homomorphisms

$$k\,\mathrm{GL}_n(k) \xrightarrow{\Phi_e} \mathrm{End}_k(L^{n,r}) \xleftarrow{\Psi_e} ek\Sigma_r e$$

such that the images of Φ_e and Ψ_e commute, so $L^{n,r}$ is a $(k\,\mathrm{GL}_n(k), ek\Sigma_r e)$-bimodule. Our main theorem is the following analogue of classical Schur-Weyl duality for this bimodule. The theorem is proved in §3.

Theorem 2.2. *Suppose that k is a field of cardinality strictly larger than r such that the characteristic of k does not divide r, and that e is a Lie idempotent in $k\Sigma_r$. Then*

$$\Psi_e(ek\Sigma_r e) = \mathrm{End}_{\mathrm{GL}_n(k)}(L^{n,r}).$$

If in addition $T^{n,r}$ is a direct sum of absolutely irreducible $k\,\mathrm{GL}_n(k)$-modules, then

$$\Phi_e(k\,\mathrm{GL}_n(k)) = \mathrm{End}_{ek\Sigma_r e}(L^{n,r}).$$

Recall that the Schur algebra over k is the algebra

$$\mathcal{S}(n,r) = \Phi(k\,\mathrm{GL}_n(k)) = \mathrm{End}_{\Sigma_r}(T^{n,r})$$

appearing in (3). Suppose that the characteristic of k is larger than r, so $|k| > r$ as well. It is well-known that in this case $k\Sigma_r$ is a split, semisimple k-algebra (see [8]) and so $\Psi(k\Sigma_r)$ is a split, semisimple k-algebra. By classical Schur-Weyl duality the

centralizer of $\Psi(k\Sigma_r)$ in $\mathrm{End}_k(T^{n,r})$ is $\mathcal{S}(n,r)$. It is easy to see that the centralizer of a split, semisimple subalgebra of $\mathrm{End}_k(T^{n,r})$ is again split, semisimple (see §3), so $\mathcal{S}(n,r)$ is a split, semisimple k-algebra. Thus, $T^{n,r}$ is a direct sum of absolutely irreducible $k\,\mathrm{GL}_n(k)$-modules and so both equalities in the theorem hold.

Corollary 2.3. *The triple $(k\,\mathrm{GL}_n(k), L^{n,r}, ek\Sigma_r e)$ satisfies Schur-Weyl duality whenever the characteristic of k is strictly larger than r.*

Since classical Schur-Weyl duality is known to hold whenever $|k| > r$ (see [2]), it is natural to ask whether $(k\,\mathrm{GL}_n(k), L^{n,r}, ek\Sigma_r e)$ satisfies Schur-Weyl duality whenever $|k| > r$ and the characteristic of k does not divide r. Based on small rank examples, it seems likely that this is indeed the case. As a step in this direction, in §4 it is shown that the second equality in the theorem is equivalent to a statement about intertwining operators between certain transitive permutation representations arising from Young subgroups of Σ_r.

Assume for a moment that $n \geq r$. Then $T^{n,r}$ is a faithful $k\Sigma_r$-module and the Schur functor $\mathfrak{f}\colon M \mapsto \epsilon M$ from left $\mathcal{S}(n,r)$-modules to left $k\Sigma_r$-modules may be defined, where $\epsilon \in \mathcal{S}(n,r)$ is an idempotent that projects $T^{n,r}$ onto its $(1^r, 0^{n-r})$-weight space. By [7, (6.3d)] we have that $\mathfrak{f}(T^{n,r}) = \epsilon T^{n,r}$ is isomorphic to the left regular $k\Sigma_r$-module ${}_{k\Sigma_r}k\Sigma_r$ and so (3) takes the form

$$\mathcal{S}(n,r) = \mathrm{End}_{k\Sigma_r}(T^{n,r}) \tag{4}$$

and

$$\mathrm{End}_{k\Sigma_r}(\mathfrak{f}(T^{n,r})) = \mathrm{End}_{k\Sigma_r}({}_{k\Sigma_r}k\Sigma_r) \cong \mathrm{End}_{\mathcal{S}(n,r)}(T^{n,r}). \tag{5}$$

Assume further that the characteristic of k does not divide r, so that a Lie idempotent e exists. Because $L^{n,r} = T^{n,r}e$ is a $GL_n(k)$-stable subspace of $T^{n,r}$, it has a natural $\mathcal{S}(n,r)$-module structure. The Lie module,

$$\mathrm{Lie}(r) = \mathfrak{f}(L^{n,r}),$$

is the $k\Sigma_r$-module obtained by applying the Schur functor to $L^{n,r}$. Then

$$\mathfrak{f}(L^{n,r}) = \epsilon L^{n,r} = \epsilon(T^{n,r}e) = (\epsilon T^{n,r})e \cong k\Sigma_r e,$$

and so the Lie module is isomorphic to the left $k\Sigma_r$-module $k\Sigma_r e$. Let $\overline{\mathcal{S}(n,r)} = \Phi_e(k\,\mathrm{GL}_n(k))$. If $\varphi \in \mathcal{S}(n,r)$, then φ is Σ_r-equivariant and so $\varphi(L^{n,r}) \subseteq L^{n,r}$. Hence restriction defines an algebra homomorphism from $\mathcal{S}(n,r)$ to $\mathrm{End}_{ek\Sigma_r e}(L^{n,r})$ with image equal to $\overline{\mathcal{S}(n,r)}$. With this notation, Theorem 2.2 immediately implies the following analogue of classical Schur-Weyl duality expressed by (4) and (5) above, with the r^{th} graded piece of the free associative algebra on n letters replaced by the r^{th} graded piece of the free Lie algebra on n letters, and with the left regular $k\Sigma_r$-module replaced by the r^{th} Lie module.

Corollary 2.4. *Suppose that $|k| > r$, the characteristic of k does not divide r, and $n \geq r$. Then*

$$\overline{\mathcal{S}(n,r)} = \mathrm{End}_{ek\Sigma_r e}(L^{n,r})$$

and

$$\mathrm{End}_{ek\Sigma_r e}(\mathfrak{f}(L^{n,r})) = \mathrm{End}_{ek\Sigma_r e}(\mathrm{Lie}(r)) \cong \mathrm{End}_{\overline{\mathcal{S}(n,r)}}(L^{n,r}),$$

where e is any Lie idempotent.

Returning to the case of general n and r, suppose the field k contains a primitive r^{th} root of unity ζ. Then the right ideal $ek\Sigma_r$ and the algebra $ek\Sigma_r e$ arise in a surprisingly different context. To describe this connection further, fix an r-cycle γ in Σ_r and let $\Gamma = \langle\gamma\rangle$. Let $f = (1/r)\sum_{i=1}^{r}\zeta^{-i}\gamma^i$. Then f is the primitive idempotent in $k\Gamma$ corresponding to a faithful character of Γ. The right ideal $fk\Sigma_r$ of $k\Sigma_r$ affords the induced representation $\operatorname{Ind}_\Gamma^{\Sigma_r}\zeta$, and the subalgebra $fk\Sigma_r f$ is isomorphic to the endomorphism algebra of the induced module $\operatorname{Ind}_\Gamma^{\Sigma_r}\zeta$. There is a Lie idempotent κ, the Klyachko idempotent (see §4), such that $e\kappa = e$, $\kappa f = f$, and $f\kappa = \kappa$. It follows that

$$ek\Sigma_r \cong \kappa k\Sigma_r = fk\Sigma_r \quad \text{and so} \quad ek\Sigma_r e \cong fk\Sigma_r f \cong \operatorname{End}_{k\Sigma_r}(\operatorname{Ind}_\Gamma^{\Sigma_r}\zeta).$$

On the other hand, suppose that $k = \mathbb{C}$ and let M denote the subset of $\mathbb{C}^n$ consisting of vectors with distinct coordinates. Then M is the complement of the union of the hyperplanes in the braid arrangement on r strands. Arnold [1] has described the cohomology ring $H^*(M)$. The group Σ_r acts on M by permuting the coordinates and hence acts on the cohomology spaces $H^p(M)$. Lehrer and Solomon [10] have described these representations of Σ_r as direct sums of representations induced from linear characters of centralizers. A special case is the r-cycle γ and its centralizer Γ. In this case, it follows from the results in [4, §5] that the representation of Σ_r afforded by the highest non-vanishing cohomology space $H^{r-1}(M)$ is isomorphic to the representation afforded by $\mathbb{C}_{\text{sgn}} \otimes f\mathbb{C}\Sigma_r \cong \mathbb{C}_{\text{sgn}} \otimes e\mathbb{C}\Sigma_r$, where $\mathbb{C}_{\text{sgn}}$ denotes the sign representation of Σ_r.

3. Generalized Schur-Weyl duality

We will now prove Theorem 2.2. It turns out that our result is a special case of a more general result, as formulated below.

Suppose that A and B are k-algebras and V is an (A, B)-bimodule. Then V is a left B^{op}-module and there are k-algebra homomorphisms

$$A \xrightarrow{\Phi} \operatorname{End}_k(V) \xleftarrow{\Psi} B^{\text{op}}\ , \tag{6}$$

where $\Phi(a)(v) = av$ and $\Psi(b)(v) = vb$ for $a \in A$, $v \in V$, and $b \in B$. Assume that the triple (A, V, B) satisfies Schur-Weyl duality, so

$$\Phi(A) = \operatorname{End}_B(V) \quad \text{and} \quad \Psi(B) = \operatorname{End}_A(V).$$

Suppose that e in B is an idempotent such that $Ve \neq 0$. Clearly Ve is an (A, eBe)-bimodule and we can ask under what conditions (A, Ve, eBe) satisfies Schur-Weyl duality. In this situation, the commuting actions induce k-algebra homomorphisms

$$A \xrightarrow{\Phi_e} \operatorname{End}_k(Ve) \xleftarrow{\Psi_e} (eBe)^{\text{op}}$$

such that

$$\Phi_e(A) \subseteq \operatorname{End}_{eBe}(Ve) \quad \text{and} \quad \Psi_e(eBe) \subseteq \operatorname{End}_A(Ve). \tag{7}$$

We wish to find conditions under which the above inclusions are equalities; that is, we wish to prove that (A, Ve, eBe) satisfies Schur-Weyl duality under appropriate hypotheses. That the second inclusion in (7) is an equality is an easy general fact, requiring no additional hypothesis.

Lemma 3.1. *Suppose that (A, V, B) satisfies Schur-Weyl duality, e is an idempotent in B such that $Ve \neq 0$, and $\Psi_e \colon eBe \to \mathrm{End}_k(Ve)$ is the k-algebra homomorphism induced by the right eBe-module structure on Ve. Then*

$$\Psi_e(eBe) = \mathrm{End}_A(Ve).$$

Proof. Set $\pi = \Psi(e)$. Then $\pi(v) = ve$ for v in V, π is an idempotent in $\mathrm{End}_A(V)$, and Ve is the image of π.

Suppose φ is in $\mathrm{End}_k(V)$. Then $\pi\varphi\pi(Ve) \subseteq Ve$. We denote the restriction of $\pi\varphi\pi$ to Ve by $\pi\varphi\pi|_{Ve}$. Then $\pi\varphi\pi|_{Ve}$ is in $\mathrm{End}_k(Ve)$. Define

$$\Pi \colon \mathrm{End}_k(V) \to \mathrm{End}_k(Ve) \quad \text{by} \quad \Pi(\varphi) = \pi\varphi\pi|_{Ve}.$$

Clearly Ve is an A-submodule of V. If φ is A-linear, then so is $\pi\varphi\pi|_{Ve}$. Therefore, $\Pi(\mathrm{End}_A(V)) \subseteq \mathrm{End}_A(Ve)$. The A-module decomposition $V \cong Ve \oplus V(1-e)$ of V determines a canonical decomposition

$$\begin{aligned} &\mathrm{End}_A(V) \cong \\ &\quad \mathrm{End}_A(Ve) \oplus \mathrm{Hom}_A(Ve, V(1-e)) \oplus \mathrm{Hom}_A(V(1-e), Ve) \oplus \mathrm{End}_A(V(1-e)) \end{aligned}$$

under which the linear map Π is identified with the projection onto $\mathrm{End}_A(Ve)$. In particular,

$$\Pi(\mathrm{End}_A(V)) = \mathrm{End}_A(Ve). \tag{8}$$

It is straightforward to check that $\Pi\Phi = \Phi_e$ and so we can extend (6) to a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[dr, "\Phi_e"'] & \mathrm{End}_k(V) \arrow[d, "\Pi"] & B \arrow[l, "\Psi"'] \arrow[d, "\Pi_e"] \\ & \mathrm{End}_k(Ve) & eBe \arrow[l, "\Psi_e"] \end{tikzcd}$$

of k-linear maps, where $\Pi_e \colon B \to eBe$ is given by $\Pi_e(b) = ebe$. Thus,

$$\Psi_e(eBe) = \Psi_e\Pi_e(B) = \Pi\Psi(B) = \Pi(\mathrm{End}_A(V)) = \mathrm{End}_A(Ve),$$

where the penultimate equality follows from the assumption that (A, V, B) satisfies Schur-Weyl duality, and the final equality follows from (8). □

The semisimple case. Showing that the containment $\Phi_e(A) \subseteq \mathrm{End}_{eBe}(Ve)$ is an equality is not so easy. We consider first the case when $\Phi(A)$ is a split, semisimple k-algebra.

Precisely, assume that V is a finite dimensional k-vector space, and suppose that (A, V, B) satisfies Schur-Weyl duality and that V is a completely reducible A-module whose irreducible constituents are absolutely irreducible. In this case we show that if e is an idempotent in B with $Ve \neq 0$, then (A, Ve, eBe) satisfies Schur-Weyl duality and both the eBe-module structure of Ve and the algebra structure of $\Psi_e(eBe)$ are determined by the A-module structure of V. The argument is based on a version of the double centralizer theorem in the form given below. We include a sketch of the proof in order to establish notation and because of the lack of a suitable reference.

Theorem 3.2. *Suppose k is a field, V is a finite dimensional k-vector space, and X is a subalgebra of $\operatorname{End}_k(V)$ such that V is a completely reducible X-module whose irreducible constituents are absolutely irreducible. Define $Y = \operatorname{End}_X(V)$ and suppose $\{L_1, \dots, L_p\}$ is a complete set of non-isomorphic, irreducible X-modules. For $1 \leq i \leq p$ define $M_i = \operatorname{Hom}_X(L_i, V)$. Then the following statements hold.*

(1) *X and Y are split, semisimple k algebras and $\{M_1, \dots, M_p\}$ is a complete set of non-isomorphic, irreducible Y-modules.*
(2) *The natural evaluation map $\bigoplus_{i=1}^p L_i \otimes_k M_i \to V$ is an (X, Y^{op})-bimodule isomorphism.*
(3) *$X = \operatorname{End}_Y(V)$ and the triple (X, V, Y^{op}) satisfies Schur-Weyl duality.*

Proof. It follows from [3, (3.31)] that X is semisimple and that each L_i occurs as an irreducible constituent of V. Let $V_1, \dots, V_p$ be the homogeneous components of V where $V_i \cong L_i^{m_i}$, so $V \cong \bigoplus_{i=1}^p V_i \cong \bigoplus_{i=1}^p L_i^{m_i}$. Set $\dim L_i = l_i$. Then $X \cong \bigoplus_{i=1}^p M_{l_i}(k)$ is a split, semisimple k-algebra.

Now let $Y = \operatorname{End}_X(V) \cong \bigoplus_{i=1}^p \operatorname{End}_X(V_i)$ be the centralizer of X in $\operatorname{End}_k(V)$. For $1 \leq i \leq p$ set $Y_i = \operatorname{End}_X(V_i)$, $M_i' = \operatorname{Hom}_X(L_i, V_i)$, and $M_i = \operatorname{Hom}_X(L_i, V)$. Then $Y_i \cong M_{m_i}(k)$ is a simple k-algebra, $Y \cong \bigoplus_{i=1}^p M_{m_i}(k)$ is a semisimple k-algebra with the property that every irreducible Y-module is absolutely irreducible, M_i' is an irreducible Y_i-module, M_i is an irreducible Y-module on which the factor Y_i acts non-trivially, and $M_i' \cong M_i$ as Y-modules, where Y acts on M_i' via the projection to Y_i. It is straightforward to check that the natural evaluation map $\varphi_i \colon L_i \otimes_k M_i' \to V_i$ with $\varphi_i(l \otimes f) = f(l)$ is an isomorphism of (X, Y_i^{op})-bimodules, where $X \times Y_i^{\mathrm{op}}$ acts on $L_i \otimes_k M_i'$ with $(x, y) \cdot l \otimes f = x \cdot l \otimes y \circ f$, and on V_i by $(x, y) \cdot v = x(y(v))$, for all $x \in X$, $y \in Y_i$, $l \in L_i$, $f \in M_i'$, and $v \in V_i$. This proves the first statement in the theorem.

Using the isomorphisms φ_i, it is straightforward to show that the natural evaluation map $\bigoplus_{i=1}^r L_i \otimes_k M_i \to V$ is an isomorphism of (X, Y^{op})-bimodules.

It follows that as a Y-module, V is isomorphic to $\bigoplus_{i=1}^p M_i^{l_i}$. Because each M_i is absolutely irreducible we see that $\operatorname{End}_Y(V) \cong \bigoplus_{i=1}^p M_{l_i}(k)$. Finally, because $X \subseteq \operatorname{End}_Y(V)$, it follows that $X = \operatorname{End}_Y(V)$. Thus (X, V, Y^{op}) satisfies Schur-Weyl duality. ∎

Now suppose (A, V, B) satisfies Schur-Weyl duality, V is a completely reducible A-module whose irreducible constituents are absolutely irreducible, and e is an idempotent in B such that $Ve \neq 0$. Set $X = \Phi(A)$ and $Y = \Psi(B^{\mathrm{op}})$. Then X and Y are subalgebras of $\operatorname{End}_k(V)$, V is a completely reducible X-module whose irreducible constituents are absolutely irreducible, $Y = \operatorname{End}_X(V)$, and $X = \operatorname{End}_Y(V)$.

Theorem 3.3. *With the assumptions and notation above, the following statements hold.*

(1) *The subalgebra $\Psi_e(eBe)$ of $\operatorname{End}_k(Ve)$ is a split, semisimple k-algebra and $\{ M_i e \mid M_i e \neq 0 \}$ is a complete set of non-isomorphic, irreducible right $\Psi_e(eBe)$-modules.*
(2) *Ve is a completely reducible right eBe-module and $\{ M_i e \mid M_i e \neq 0 \}$ is a complete set of non-isomorphic, irreducible right eBe-modules that occur as constituents of Ve.*
(3) *The triple (A, Ve, eBe) satisfies Schur-Weyl duality.*

Proof. Set $\pi = \Psi(e)$, so π is a non-zero idempotent in Y. By Theorem 3.2 and the general theory of split, semisimple algebras, $\pi Y \pi$ is a split, semisimple k-algebra and $\{\, \pi M_i \mid \pi M_i \neq 0 \,\}$ is a complete set of non-isomorphic, irreducible $\pi Y \pi$-modules. Now $\pi(V) = Ve$ and the image of the homomorphism $\Psi_e \colon (eBe)^{\mathrm{op}} \to \mathrm{End}_k(Ve)$ coincides with the image of the natural homomorphism $\Psi_\pi \colon \pi Y \pi \to \mathrm{End}_k(\pi(V))$. Moreover, by definition $\pi M_i = M_i e$ for $1 \leq i \leq p$. Therefore, $\Psi_e(eBe)$ is a split, semisimple k-algebra and $\{\, M_i e \mid M_i e \neq 0 \,\}$ is a complete set of non-isomorphic, irreducible right $\Psi_e(eBe)$-modules.

The algebra A acts on $\bigoplus_{i=1}^p L_i \otimes_k M_i$ through its left action on each L_i, and the algebra B acts on $\bigoplus_{i=1}^p L_i \otimes_k M_i$ through its right action on each M_i. Therefore, the (A,B)-bimodule isomorphism $V \cong \bigoplus_{i=1}^p L_i \otimes_k M_i$ induces an (A, eBe)-bimodule isomorphism

$$Ve \cong \bigoplus_{M_i e \neq 0} L_i \otimes_k M_i e. \tag{9}$$

If $M_i e \neq 0$, then $M_i e$ is an absolutely irreducible $\Psi_e(eBe)$-module and hence an absolutely irreducible eBe-module. The bimodule isomorphism (9) induces an isomorphism of right eBe-modules

$$Ve \cong \bigoplus_{M_i e \neq 0} (M_i e)^{\dim L_i},$$

which proves the second statement in the theorem.

Finally, set $X_1 = \Phi_e(A)$. Then Ve is a completely reducible X_1-module and so by Theorem 3.2, if $Y_1 = \mathrm{End}_{X_1}(Ve)$, then $X_1 = \mathrm{End}_{Y_1}(Ve)$. Clearly $\mathrm{End}_{X_1}(Ve) = \mathrm{End}_A(Ve)$ and by Lemma 3.1, $\mathrm{End}_A(Ve) = \Psi_e(eBe)$. Moreover, $\Phi_e(A) = X_1 = \mathrm{End}_{Y_1}(Ve) = \mathrm{End}_{eBe}(Ve)$, and hence (A, Ve, eBe) satisfies Schur-Weyl duality, as claimed. □

We can now complete the proof of Theorem 2.2.

Proof of Theorem 2.2. Suppose that k is a field such that the cardinality of k is larger than r and the characteristic of k does not divide r. Then by classical Schur-Weyl duality, the triple $(k\,\mathrm{GL}_n(k), T^{n,r}, k\Sigma_r)$ satisfies Schur-Weyl duality. Moreover, $L^{n,r} = T^{n,r}e$. Thus, it follows from Lemma 3.1 that $\Psi_e(ek\Sigma_r e) = \mathrm{End}_{\mathrm{GL}_n(k)}(L^{n,r})$.

If in addition $T^{n,r}$ is a direct sum of absolutely irreducible $k\,\mathrm{GL}_n(k)$-modules, then it follows from Theorem 3.3 that $(k\,\mathrm{GL}_n(k), L^{n,r}, ek\Sigma_r e)$ satisfies Schur-Weyl duality, and so in particular, $\Phi_e(k\,\mathrm{GL}_n(k)) = \mathrm{End}_{ek\Sigma_r e}(L^{n,r})$. This completes the proof of the theorem. □

Idempotents. We now return to the general situation considered at the beginning of this section where the triple (A, V, B) satisfies Schur-Weyl duality and $e \in B$ is an idempotent with $Ve \neq 0$. We give various conditions that are equivalent to the assertion that the triple (A, Ve, eBe) satisfies Schur-Weyl duality.

To start, consider the k-algebra homomorphisms

$$\Phi' \colon A \to \mathrm{End}_B(V) \quad \text{and} \quad \Phi'_e \colon A \to \mathrm{End}_{eBe}(Ve)$$

induced by Φ and Φ_e, respectively. Because (A, V, B) satisfies Schur-Weyl duality, Φ' is surjective, and by Lemma 3.1, (A, Ve, eBe) satisfies Schur-Weyl duality if and only if Φ'_e is surjective.

Suppose φ is in $\mathrm{End}_B(V)$. Then $\varphi(Ve) = \varphi(V)e \subseteq Ve$ and the restriction of φ to Ve induces an eBe-linear homomorphism $\bar{\varphi}\colon Ve \to Ve$. Define

$$\Theta_e\colon \mathrm{End}_B(V) \to \mathrm{End}_{eBe}(Ve) \quad \text{by} \quad \Theta_e(\varphi) = \bar{\varphi}.$$

It follows immediately from the definitions that $\Theta_e\Phi' = \Phi'_e$. Because Φ' is surjective, it then follows that Φ'_e is surjective if and only if Θ_e is surjective. Clearly Θ_e is surjective if and only if every eBe-linear endomorphism of Ve extends to a B-linear endomorphism of V. This proves the next lemma.

Lemma 3.4. *Suppose that (A, V, B) satisfies Schur-Weyl duality and e is an idempotent in B such that $Ve \neq 0$. Then the following are equivalent.*

1. *The triple (A, Ve, eBe) satisfies Schur-Weyl duality.*
2. *Every eBe-linear endomorphism of Ve extends to a B-linear endomorphism of V.*

Note that the second condition in the lemma depends only on B and the right B-module structure on V, and not on the algebra A. This observation can be used to replace the idempotent e by any suitably equivalent idempotent f, as we now explain. Suppose f is an idempotent in B such that

$$ef = f \quad \text{and} \quad fe = e.$$

Then the maps $\rho_f\colon Ve \to Vf$ and $\rho_e\colon Vf \to Ve$ given by $\rho_f(x) = xf$ and $\rho_e(x) = xe$ are mutual inverses. It is straightforward to check that

$$\Xi\colon \mathrm{End}_{eBe}(Ve) \to \mathrm{End}_{fBf}(Vf) \quad \text{by} \quad \Xi(\varphi) = \rho_f\varphi\rho_e$$

is an algebra isomorphism, with inverse $\Xi^{-1}(\psi) = \rho_e\psi\rho_f$ for ψ in $\mathrm{End}_{fBf}(Vf)$. It is also straightforward to check that $\Xi\,\Theta_e = \Theta_f$. This proves the next lemma.

Lemma 3.5. *With the notation as above, Θ_e is surjective if and only if Θ_f is surjective.*

The next theorem follows from Lemma 3.4 and Lemma 3.5.

Theorem 3.6. *Suppose that (A, V, B) satisfies Schur-Weyl duality, and e and f are idempotents in B such that $Ve \neq 0$, $ef = f$, and $fe = e$. Then the following are equivalent.*

1. *(A, Ve, eBe) satisfies Schur-Weyl duality.*
2. *Every eBe-linear endomorphism of Ve extends to a B-linear endomorphism of V.*
3. *(A, Vf, fBf) satisfies Schur-Weyl duality.*
4. *Every fBf-linear endomorphism of Vf extends to a B-linear endomorphism of V.*

4. Complements

In this section we use the results in the previous section first to investigate the commuting algebra of the $\mathrm{GL}_n(k)$-action on $L^{n,r}$ when everything is semisimple, and second to characterize when the triple $(k\,\mathrm{GL}_n(k), L^{n,r}, ek\Sigma_r e)$ satisfies Schur-Weyl duality, in terms of certain permutation representations of Σ_r. Throughout this section we assume that $|k| > r$, that the characteristic of k does not divide r, and that k contains a primitive r^{th} root of unity ζ.

If $e = \sum_{\sigma\in\Sigma_r} a_\sigma\sigma$ is any idempotent in $k\Sigma_r$, then a result of Littlewood (see [3, Exercise 9.16] shows that the character of the right $k\Sigma_r$-module $ek\Sigma_r$, evaluated at

a permutation τ, is the sum $\sum_{\sigma\in\mathcal{C}} a_\sigma$, where $\mathcal{C}$ is the conjugacy class of τ. When e is a Lie idempotent, Garsia [5, Theorem 5.2] gives a formula for the sums $\sum_{\sigma\in\mathcal{C}} a_\sigma$. This formula does not depend on the choice of Lie idempotent. Therefore, up to isomorphism, the right ideal $ek\Sigma_r$ does not depend on the choice of e and so the following lemma follows from Lemma 3.4.

Lemma 4.1. *Suppose e and e' are Lie idempotents. Then $(k\,\mathrm{GL}_n(k), L^{n,r}, ek\Sigma_r e)$ satisfies Schur-Weyl duality if and only if $(k\,\mathrm{GL}_n(k), L^{n,r}, e'k\Sigma_r e')$ does.*

By the lemma, we may choose e to be any convenient Lie idempotent. In this section we use a Lie idempotent found by Klyachko.

Given a permutation σ, an integer i is a descent of σ if $\sigma(i) > \sigma(i+1)$. Let $\mathcal{D}(\sigma)$ denote the set of descents of σ. By definition, the major index of σ is

$$\mathrm{maj}(\sigma) = \sum_{i\in\mathcal{D}(\sigma)} i.$$

The Klyachko idempotent is the element

$$\kappa = \frac{1}{r}\sum_{\sigma\in\Sigma_r} \zeta^{\mathrm{maj}(\sigma)}\sigma$$

in $k\Sigma_r$. Klyachko [9] has shown that κ is a Lie idempotent. Furthermore, if γ is any fixed r-cycle in Σ_r and we define

$$f = \frac{1}{r}\sum_{i=1}^{r} \zeta^{-i}\gamma^i \tag{10}$$

as in §2, then $\kappa f = f$ and $f\kappa = \kappa$ (see [11, §8.4]). Set

$$H = fk\Sigma_r f.$$

Then for any Lie idempotent e we have

$$ek\Sigma_r e \cong \kappa k\Sigma_r\kappa \cong H \cong \mathrm{End}_{k\Sigma_r}(\mathrm{Ind}_\Gamma^{\Sigma_r}\zeta) \tag{11}$$

where $\Gamma = \langle\gamma\rangle$ is the subgroup generated by γ.

The semisimple case. Now assume that the characteristic of k is greater than r, so $k\Sigma_r$ and H are split, semisimple k-algebras, and consider the commuting algebra $\mathrm{End}_{\mathrm{GL}_n(k)}(L^{n,r})$. By Corollary 2.3, the triple $(k\,\mathrm{GL}_n(k), L^{n,r}, \kappa k\Sigma_r\kappa)$ satisfies Schur-Weyl duality and so by Theorem 3.6, $(k\,\mathrm{GL}_n(k), T^{n,r}f, H)$ does as well. Note that right multiplication by κ defines a $\mathrm{GL}_n(k)$-equivariant isomorphism between $T^{n,r}f$ and $L^{n,r}$ that intertwines the right actions of H and $\kappa k\Sigma_r\kappa$, and that by (11),

$$\mathrm{End}_{\mathrm{GL}_n(k)}(L^{n,r}) \cong H \cong \mathrm{End}_{k\Sigma_r}(\mathrm{Ind}_\Gamma^{\Sigma_r}\zeta).$$

In the following, we consider the algebra H instead of $\mathrm{End}_{\mathrm{GL}_n(k)}(L^{n,r})$.

Recall that a partition is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that (1) $\lambda_1 \geq \lambda_2 \geq \dots$ and (2) $\lambda_i \neq 0$ for only finitely many i. If $\lambda_i > 0$, then λ_i is a part of λ. Define $\ell(\lambda)$ to be the number of parts of λ. If $\sum_{i>0}\lambda_i = r$, then say λ is a partition of r and write $\lambda \vdash r$. When $\ell(\lambda) = a$ we generally abuse notation and write $\lambda = (\lambda_1, \dots, \lambda_a)$ instead of $\lambda = (\lambda_1, \dots, \lambda_a, 0, \dots)$.

For a partition λ of r with at most n parts let V^λ be the irreducible representation of $\mathrm{GL}_n(k)$ with highest weight λ and let S^λ be the Specht module indexed by λ. For example, if $\lambda = (r)$, then V^λ is the natural module for $\mathrm{GL}_n(k)$ and S^λ is the

trivial representation of Σ_r, and if all the parts of λ are equal to 1, then V^λ is the determinant representation of $\mathrm{GL}_n(k)$ and S^λ is the sign representation of Σ_r. Semisimplicity implies (see e.g. [6, Proposition 3.3.2]) that there is an isomorphism of $(k\,\mathrm{GL}_n(k), k\Sigma_r)$-bimodules

$$T^{n,r} \cong \bigoplus_{\substack{\lambda\vdash r\\ \ell(\lambda)\le n}} V^\lambda \otimes S^\lambda. \tag{12}$$

Applying the map ρ_f to (12) gives

$$T^{n,r}f \cong \bigoplus_{\substack{\lambda\vdash r\\ \ell(\lambda)\le n}} V^\lambda \otimes S^\lambda f$$

where $\{\, S^\lambda f \mid S^\lambda f \neq 0\,\}$ is a set of non-isomorphic, irreducible, right H-modules. For a partition λ of r with $S^\lambda f \neq 0$, let H_λ denote the minimal two-sided ideal of H with the property that $S^\lambda f H_\lambda \neq 0$.

The decomposition of $k\Sigma_r f$ into irreducible constituents is given in [11, Chapter 8] and [5]. This decomposition determines the algebra structure of H as follows.

Let $\mathrm{S}YT$ denote the set of standard Young tableaux with r boxes. For a partition λ of r let $\mathrm{S}YT^\lambda$ be the set of standard Young tableaux with shape λ. Suppose t is a standard Young tableau. An integer i is a descent of t if $i+1$ occurs in a lower row of t than i. Let $\mathcal{D}(t)$ denote the set of descents of t. The major index of t is

$$\mathrm{maj}(\mathsf{t}) = \sum_{i\in\mathcal{D}(t)} i.$$

Define

$$\mathrm{S}YT^\lambda_{\equiv 1} = \{\, \mathsf{t}\in \mathrm{S}YT^\lambda \mid \mathrm{maj}(\mathsf{t}) \equiv 1 \bmod r\,\}.$$

Then the multiplicity of S^λ in $k\Sigma_r f$ is $|\mathrm{S}YT^\lambda_{\equiv 1}|$. Thus,

(H1) the simple H-modules are parametrized by the set of partitions λ of r for which $|\mathrm{S}YT^\lambda_{\equiv 1}| \neq \emptyset$,
(H2) the dimension of $S^\lambda f$ is $|\mathrm{S}YT^\lambda_{\equiv 1}|$,
(H3) the dimension of H_λ is $|\mathrm{S}YT^\lambda_{\equiv 1}|^2$, and
(H4) $\dim H = \sum_{\lambda\vdash r} |\mathrm{S}YT^\lambda_{\equiv 1}|^2$.

Obviously $|\mathrm{S}YT^\lambda_{\equiv 1}|$ depends on only the integer r, and not the field k, so one might hope that statements (H1)–(H4) hold whenever the characteristic of k does not divide r.

The partitions λ such that $S^\lambda f \neq 0$ have been determined by Klyachko, as follows.

Theorem 4.2. *Suppose that λ is a partition of r. Then there is a standard Young tableau with shape λ and major index congruent to 1 mod r if and only if λ is not equal to $(1,1,\dots,1)$, (r), $(2,2)$ (in case $r=4$), or $(2,2,2)$ (in case $r=6$).*

We can use the Robinson-Schensted correspondence between Σ_r and the set of pairs of standard Young tableaux with the same shape to obtain formulas for the dimensions of H_λ and H in terms of permutations instead of tableaux. Let $P\colon \Sigma_r \to \mathrm{S}YT$ be the map given by the Schensted (row) insertion algorithm. Then the Robinson-Schensted correspondence is given by the assignment $\sigma \mapsto (P(\sigma), P(\sigma^{-1}))$.

Proposition 4.3. *Suppose that the characteristic of k is greater than r and that k contains a primitive r^{th} root of unity. Then*

$$\dim H_\lambda = |\{\, \sigma \in \Sigma_r \mid P(\sigma) \in \mathrm{SY}T^\lambda,\ \mathrm{maj}(\sigma) \equiv 1 \bmod r,\ \mathrm{maj}(\sigma^{-1}) \equiv 1 \bmod r \,\}|$$

and

$$\dim H = |\{\, \sigma \in \Sigma_r \mid \mathrm{maj}(\sigma) \equiv 1 \bmod r,\ \mathrm{maj}(\sigma^{-1}) \equiv 1 \bmod r \,\}|.$$

Thus, the dimension of H is the number of permutations σ such that σ and σ^{-1} both have major index congruent to 1 *modulo r.*

Proof. Clearly, the preimage of $\mathrm{SY}T^\lambda_{\equiv 1} \times \mathrm{SY}T^\lambda_{\equiv 1}$ in Σ_r is

$$\{\, \sigma \in \Sigma_r \mid P(\sigma) \in \mathrm{SY}T^\lambda,\ \mathrm{maj}(P(\sigma)) \equiv 1 \bmod r,\ \mathrm{maj}(P(\sigma^{-1})) \equiv 1 \bmod r \,\}.$$

A standard property of the row insertion algorithm is that $\mathcal{D}(\sigma) = \mathcal{D}(P(\sigma^{-1}))$ and so $\mathrm{maj}(\sigma) = \mathrm{maj}(P(\sigma^{-1}))$. Thus, the preimage of $\mathrm{SY}T^\lambda_{\equiv 1} \times \mathrm{SY}T^\lambda_{\equiv 1}$ is the set of permutations σ such that $P(\sigma)$ has shape λ, $\mathrm{maj}(\sigma) \equiv 1 \bmod r$, and $\mathrm{maj}(\sigma^{-1}) \equiv 1 \bmod r$. Thus the proposition follows (H3) and (H4). $\square$

The non-semisimple case. Now we return to the general situation, assuming only that the characteristic of k does not divide r, so $T^{n,r}$ is not necessarily completely reducible. Our goal is to characterize when the triple $(k\,\mathrm{GL}_n(k), L^{n,r}, ek\Sigma_r e)$ satisfies Schur-Weyl duality in terms of certain permutation representations of Σ_r. We continue to use the idempotent $f \in k\Gamma$ defined in (10).

It was shown in Theorem 3.6 that the triple $(k\,\mathrm{GL}_n(k), L^{n,r}, \kappa k\Sigma_r\kappa)$ satisfies Schur-Weyl duality if and only if the restriction homomorphism

$$\Theta_f \colon \mathrm{End}_{k\Sigma_r}(T^{n,r}) \to \mathrm{End}_H(T^{n,r}f)$$

is surjective. To streamline the notation, set $\Theta = \Theta_f$. To find conditions under which Θ is surjective we need to consider some standard notation and constructions.

Denote the set $\{1, 2, \dots, n\}$ simply by $[n]$. The group Σ_r acts on the set $[n]^r$ on the right by $(a\sigma)_j = a_{\sigma(j)}$ for a in $[n]^r$ and σ in Σ_r. Let $\{v_1, \dots, v_r\}$ be the standard basis of V. For $a = (a_1, \dots, a_r)$ in $[n]^r$ define

$$v_a = v_{a_1} \otimes \cdots \otimes v_{a_r}$$

in $T^{n,r}$. Then

$$\mathcal{B} = \{\, v_a \mid a \in [n]^r \}$$

is a k-basis of $T^{n,r}$. Clearly $v_a\sigma = v_{a\sigma}$ for a in $[n]^r$ and σ in Σ_r, so $\mathcal{B}$ is a Σ_r-stable subset of $T^{n,r}$.

Technically, an element a in $[n]^r$ is a function $a : [r] \to [n]$. In particular, if σ is in Σ_r, then $a \circ \sigma \colon [r] \to [n]$. Thus, the right action of Σ_r on $[n]^r$ is simply the natural right action of Σ_r of the set of functions $[r] \to [n]$ given by $(a, \sigma) \mapsto a \circ \sigma$. Similarly, the group Σ_n acts naturally on $[n]^r$ on the left. Namely, if $a = (a_1, a_2, \dots, a_r)$ is in $[n]^r$ and τ is in Σ_n, then

$$\tau((a_1, a_2, \dots, a_r)) = (\tau(a_1), \tau(a_2), \dots, \tau(a_r)).$$

Clearly the left Σ_n-action and the right Σ_r-action commute.

For a in $[n]^r$, define the content of a to be the n-tuple

$$\mathrm{ct}(a) = (|a^{-1}(1)|, |a^{-1}(2)|, \dots, |a^{-1}(n)|).$$

Then $\mathrm{ct}(a) = (m_1, m_2, \dots, m_n)$, where m_j is the multiplicity of j in a. It is easy to see that ct is an orbit map for the right action of Σ_r on $[n]^r$. In other words,

$\mathrm{ct}(a) = \mathrm{ct}(b)$ if and only if there is a σ in Σ_r such that $b = a\sigma$. Define $\Lambda(n,r)$ to be the image of ct. Then

$$\Lambda(n,r) = \{ (m_1, m_2, \dots, m_n) \in \mathbb{N}^n \mid m_1 + m_2 + \cdots + m_n = r \},$$

and so $\Lambda(n,r)$ is the set of compositions of r into at most n parts, with parts of length zero allowed. It is well-known and straightforward to check that $\Lambda(n,r)$ may be identified with the set of weights of the group of diagonal matrices in $\mathrm{GL}_n(k)$ acting on the space $T^{n,r}$. Elements of $\Lambda(n,r)$ are thus referred to as "weights." In the following, we fix n and r and set $\Lambda = \Lambda(n,r)$.

For a weight α in Λ, define

$$[n]^r_\alpha = \mathrm{ct}^{-1}(\alpha) = \{ a \in [n]^r \mid \mathrm{ct}(a) = \alpha \},$$

and define the weight space $T^{n,r}_\alpha$ of $T^{n,r}$ by

$$T^{n,r}_\alpha = \mathrm{span}\{ v_a \in \mathcal{B} \mid a \in [n]^r_\alpha \} = \mathrm{span}\{ v_a \in \mathcal{B} \mid \mathrm{ct}(a) = \alpha \}.$$

Because $\mathrm{ct}(a\sigma) = \mathrm{ct}(a)$ for all $a \in [n]^r$ and $\sigma \in \Sigma_r$, it follows that for each weight α, $T^{n,r}_\alpha$ is a right Σ_r-submodule of $T^{n,r}$ and that

$$T^{n,r} \cong \bigoplus_{\alpha \in \Lambda} T^{n,r}_\alpha \tag{13}$$

as right Σ_r-modules.

Now consider the subspace $T^{n,r}f$ of $T^{n,r}$. Because the left action of $\mathrm{GL}_n(k)$ on $T^{n,r}$ commutes with the right action of Σ_r, it follows that $T^{n,r}f$ is a $\mathrm{GL}_n(k)$-stable subspace of $T^{n,r}$, and hence that for $\alpha \in \Lambda$, the α weight space of $T^{n,r}f$ is equal to $T^{n,r}_\alpha f$. Then $T^{n,r}_\alpha f$ is an H-submodule of $T^{n,r}f$ and

$$T^{n,r}f \cong \bigoplus_{\alpha \in \Lambda} T^{n,r}_\alpha f \tag{14}$$

as right H-modules.

Recall that $\Theta\colon \mathrm{End}_{\Sigma_r}(T^{n,r}) \to \mathrm{End}_H(T^{n,r}f)$ is given by $\Theta(\varphi) = \bar{\varphi}$, where $\bar{\varphi}\colon T^{n,r} \to T^{n,r}f$ is the restriction of φ. The decompositions (13) and (14) induce isomorphisms of k-vector spaces

$$\mathrm{End}_{\Sigma_r}(T^{n,r}) \cong \bigoplus_{\alpha,\beta \in \Lambda} \mathrm{Hom}_{\Sigma_r}(T^{n,r}_\alpha, T^{n,r}_\beta) \tag{15}$$

and

$$\mathrm{End}_H(T^{n,r}f) \cong \bigoplus_{\alpha,\beta \in \Lambda} \mathrm{Hom}_H(T^{n,r}_\alpha f, T^{n,r}_\beta f). \tag{16}$$

Suppose α and β are in Λ. If ψ is in $\mathrm{Hom}_{\Sigma_r}(T^{n,r}_\alpha, T^{n,r}_\beta)$, then $\psi(T^{n,r}_\alpha f) \subseteq T^{n,r}_\beta f$ and the restriction of ψ to $T^{n,r}_\alpha$ is in $\mathrm{Hom}_H(T^{n,r}_\alpha f, T^{n,r}_\beta f)$. Define

$$\Theta^\alpha_\beta\colon \mathrm{Hom}_{\Sigma_r}(T^{n,r}_\alpha, T^{n,r}_\beta) \to \mathrm{Hom}_H(T^{n,r}_\alpha f, T^{n,r}_\beta f) \quad \text{by} \quad \Theta^\alpha_\beta(\psi) = \bar{\psi},$$

where $\bar{\psi}\colon T^{n,r}_\alpha f \to T^{n,r}_\beta f$ is the restriction of ψ. The maps Θ and Θ^α_β are compatible with the decompositions (15) and (16) in the sense that the diagram

$$\begin{tikzcd}
\mathrm{End}_{\Sigma_r}(T^{n,r}) \arrow[r, "\cong"] \arrow[d, "\Theta"] & \bigoplus_{\alpha,\beta\in\Lambda} \mathrm{Hom}_{\Sigma_r}(T^{n,r}_\alpha, T^{n,r}_\beta) \arrow[d, "\bigoplus \Theta^\alpha_\beta"] \\
\mathrm{End}_H(T^{n,r}f) \arrow[r, "\cong"] & \bigoplus_{\alpha,\beta\in\Lambda} \mathrm{Hom}_H(T^{n,r}_\alpha f, T^{n,r}_\beta f)
\end{tikzcd}$$

commutes. Therefore, Θ is surjective if and only if Θ^α_β is surjective for all α and β in Λ. The next proposition thus follows from Lemma 4.1, Lemma 3.4, and Theorem 3.6.

Proposition 4.4. *Suppose e is a Lie idempotent. Then $(k\,\mathrm{GL}_n(k), L^{n,r}, ek\Sigma_r e)$ satisfies Schur-Weyl duality if and only if the maps*

$$\Theta^\alpha_\beta\colon \operatorname{Hom}_{\Sigma_r}(T^{n,r}_\alpha, T^{n,r}_\beta) \to \operatorname{Hom}_H(T^{n,r}_\alpha f, T^{n,r}_\beta f)$$

are surjections for all α and β in $\Lambda(n,r)$.

Next, suppose that α is a weight in Λ. Up to the left action of Σ_n, we may assume that $\alpha = (m_1, m_2, \ldots, m_p, 0, \ldots, 0)$ where $(m_1, m_2, \ldots, m_p)$ is a composition of r with no parts that equal zero. Let

$$\Sigma_\alpha \cong \Sigma_{m_1} \times \Sigma_{m_2} \times \cdots \times \Sigma_{m_p}$$

be the corresponding Young subgroup of Σ_r. The transitive action of Σ_r on $[n]^r_\alpha$ induces an isomorphism of right $k\Sigma_r$-modules $T^{n,r}_\alpha \cong k_\alpha \otimes_{k\Sigma_\alpha} k\Sigma_r$, where k_α is the trivial right $k\Sigma_\alpha$-module, as follows.

Given $a = (a_1, a_2, \ldots, a_r)$ in $[n]^r_\alpha$, replace the occurrences of 1 in a from left to right with $1, 2, \ldots, m_1$, then replace the occurrences of 2 in a from left to right by $m_1{+}1, m_1{+}2, \ldots, m_1{+}m_2$, and so on. Define σ_a to be the permutation given in one line notation by the resulting r-tuple. For example if $r = 8$, $\alpha = (4, 2, 2, 0, \ldots, 0)$, and $b = (2, 1, 1, 3, 2, 1, 1, 3)$, then in one line notation $\sigma_b = (5, 1, 2, 7, 6, 3, 4, 8)$; that is, $\sigma_b(1) = 5$, $\sigma_b(2) = 1$, and so on. It is easy to see that the assignment $a \mapsto \sigma_a$ defines a bijection between $[n]^r_\alpha$ and the set Σ^α of minimal length right coset representatives of Σ_α in Σ_r, and that the assignment $a \mapsto \Sigma_\alpha\sigma_a$ defines a Σ_r-equivariant bijection between $[n]^r_\alpha$ and the set of right cosets $\Sigma_\alpha\backslash\Sigma_r$. Thus, the assignment $v_a \mapsto 1 \otimes \sigma_a$ defines an isomorphism of right $k\Sigma_r$-modules

$$h_\alpha\colon T^{n,r}_\alpha \xrightarrow{\cong} k_\alpha \otimes_{k\Sigma_\alpha} k\Sigma_r.$$

To simplify the notation, set $M^\alpha = k_\alpha \otimes_{k\Sigma_\alpha} k\Sigma_r$.

Now suppose that $\alpha, \beta \in \Lambda$. The assignment $\varphi \mapsto h_\beta\varphi h_\alpha^{-1}$ defines isomorphisms of k-vector spaces

$$\operatorname{Hom}_{\Sigma_r}(T^{n,r}_\alpha, T^{n,r}_\beta) \xrightarrow{\cong} \operatorname{Hom}_{\Sigma_r}(M^\alpha, M^\beta)$$

and

$$\operatorname{Hom}_H(T^{n,r}_\alpha f, T^{n,r}_\beta f) \xrightarrow{\cong} \operatorname{Hom}_H(M^\alpha f, M^\beta f),$$

such that the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\Sigma_r}(T^{n,r}_\alpha, T^{n,r}_\beta) \arrow[r, "\Theta^\alpha_\beta"] \arrow[d, "\cong"'] & \operatorname{Hom}_H(T^{n,r}_\alpha f, T^{n,r}_\beta f) \arrow[d, "\cong"'] \\
\operatorname{Hom}_{\Sigma_r}(M^\alpha, M^\beta) \arrow[r, "\theta^\alpha_\beta"] & \operatorname{Hom}_H(M^\alpha f, M^\beta f)
\end{tikzcd}$$

commutes, where the map θ^α_β on the bottom is again given by restriction. Obviously θ^α_β is surjective if and only if θ^α_β is. Combining this observation with Proposition 4.4 we obtain the following corollary.

Corollary 4.5. *Suppose e is a Lie idempotent. Then $(k\,\mathrm{GL}_n(k), L^{n,r}, ek\Sigma_r e)$ satisfies Schur-Weyl duality if and only if the restriction maps*

$$\theta^\alpha_\beta\colon \operatorname{Hom}_{\Sigma_r}(M^\alpha, M^\beta) \to \operatorname{Hom}_H(M^\alpha f, M^\beta f)$$

are surjections for all α and β in $\Lambda(n,r)$.

Thus we arrive at the following problem.

Problem 4.6. Find a combinatorially defined basis for $\operatorname{Hom}_H(M^\alpha f, M^\beta f)$ and hence show that $\dim \operatorname{Hom}_H(M^\alpha f, M^\beta f)$ does not depend on the field k. $\square$

A solution to this problem should show that $(k\,\mathrm{GL}_n(k), L^{n,r}, ek\Sigma_r e)$ satisfies Schur-Weyl duality whenever k is a field of characteristic not dividing r that contains a primitive r^{th} root of unity.

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